On the Primitive Groups of Classes Six and Eight.[*]

By W. A. Manning.

In the *Comptes Rendus* of Dec. 23, 1872, page 1754, Jordan published the results of his investigation of the primitive groups of the first 13 classes. The calculations by which these results were reached were "long and laborious" and have not been published. In view of the fundamental importance of the primitive groups, it seems desirable that these proofs should be supplied. This was done in part by Jordan[†] when he determined all the primitive groups of class p (p a prime number), and by the present writer[‡] in the determination of the primitive groups of class $2p$, which contain a substitution of order p and degree $2p$. Netto[§] and Miller[||] have considered the groups of class 4, and their results agree with Jordan's. The primitive groups of class 6, which do not contain a substitution of order 3 on 6 letters, and the primitive groups of class 8 are the object of investigation in this paper. It is found that there are in all 14 groups of class 6 and 18 groups of class 8. Jordan's lists give 13 groups of class 6 and 15 of class 8.

The following theorem is restated in altered form from a memoir by Jordan.[¶]

Theorem I. Let A, A', ... be a complete set of conjugate substitutions of prime order p and degree qp in a primitive group G. The set A, ... generates a transitive group H. Let I be an intransitive subgroup of H generated by a certain number of the substitutions A, ... Let there be a set of intransitivity $(a, \ldots.)$ in I such that none of the substitutions A, ... transform all the letters

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† C. Jordan: *Liouville's Journal*, Ser. 2, Vol. XVII (1872), p. 363.

‡ *Transactions of the American Mathematical Society*, Vol. IV (1903), p. 351. See also "On the Groups of Class Ten," American Journal of Mathematics, Vol. XXVIII (1906), p. 226.

§ Netto: "Theory of Substitutions," Cole's translation (1892), p. 133.

|| Miller: *American Mathematical Monthly*, Vol. IX (1902), p. 63.

¶ *Crelle's Journal*, Vol. LXXIX (1874), pp. 249–253.

a of that set into other letters. Let $b, \ldots.$ be the letters of the other sets of intransitivity of I and let $c, \ldots.$ be the letters of H not displaced by I. Then in H there is always a substitution B similar to A which connects a and b transitively, and has not more than one letter c in each of its q cycles. This happens notably when the number of letters in the set a exceeds $q\,\frac{p-1}{2}$, if p is odd, or q, if $p=2$.

COROLLARY I. If $p=2$, and the number of letters $a, \ldots.$ is exactly q, the theorem clearly holds unless, perhaps, when all the substitutions which replace the letters $a, \ldots.$ by other letters are of the form $C=(a_1c_1)(a_2c_2)\ldots.(a_qc_q)$. When, in this exceptional case, G is of class $2q$, any substitution A_1, similar to A, of I is of the form $(a_1a_2)\ldots.(b_1b_2)\ldots.$ where at least half the letters are $b, \ldots..$ But $A_1CA_1C=(a_1a_2)\ldots.(c_1c_2)\ldots.$ is of degree not greater than $2q$. Hence we conclude that every substitution similar to A in I, which displaces any of the letters $a, \ldots.$, displaces all of the q letters $a, \ldots..$ This exceptional case can not arise in a group of class $2q$ unless q is even. We also note that C and A_1CA_1C displace exactly the same $2q$ letters.

COROLLARY II. Suppose that there are just $q-1$ letters $a\ldots..$ We can always find a substitution C similar to A connecting $a\ldots.$ and $b\ldots.$ and having in no cycle more than one new letter c unless *any* substitution of H, similar to A, which replaces all the letters $a\ldots.$ by other letters, has one of the four following forms:

$$
\begin{aligned}
C_1&=(a_1c_1)\ldots.(a_{q-1}c_{q-1})(c_qc_{q+1}),\\
C_2&=(a_1c_1)\ldots.(a_{q-1}c_{q-1})(b_1c_q),\\
C_3&=(a_1c_1)\ldots.(a_{q-1}c_{q-1})(b_1b_2),\\
C_4&=(a_1b_1)\ldots.(a_\lambda b_\lambda)(a_{\lambda+1}c_1)\ldots.(a_{q-1}c_{q-1-\lambda})(c_{q-\lambda}c_{q-\lambda+1}),
\end{aligned}
$$

where

$$1 \overline{\overline{<}} \lambda \overline{\overline{<}} q-1.$$

It will be shown that C_1 and C_2 never occur in H, and that C_3 occurs only when *any* substitution of I, which is similar to A, contains both b_1 and b_2 in different cycles, and displaces all the $q-1$ letters $a\ldots.$ when q is odd, or all but one when q is even.

Let $A_1=(a_1a_2)\ldots.(b_1b_2)\ldots.$ be a substitution of I similar to A. The number of letters $b\ldots.$ in A_1 is at least $q+1$. Then $A_1C_1A_1C_1$ is of degree less than $2q$ and is not the identity. If A_1 does not displace the letter b_1 of C_2, the same is true of $A_1C_2A_1C_2$. Let A_1 displace b_1. Then $A_1C_2A_1C_2$ is at most of degree

$2q+1$ and is non-regular, having one cycle of order 3 and the remaining cycles of order 2. But in a group of class u, the substitutions of degree u and $u+1$ are regular. Passing to C_3 it is clear that A_1 must contain both b_1 and b_2, and in different cycles. We write $A_1=(a_1a_2)\ldots.(b_1b')(b_2b'')$ and have

$$C_3A_1C_3=(c_1c_2)\ldots.(b_2b')(b_1b'')$$

so that

$$D=A_1C_3A_1C_3=(a_1a_2)\ldots.(c_1c_2)\ldots.(b_1b_2)(b'b'').$$

If q is odd D is of degree $2q+2$, and if q is even, D is of degree $2q$.

It is easy to show, in the same way, that in C_4 λ is greater than 2, except possibly in the case of $\lambda=2$ and $q=5$, when there may be a substitution of degree 10 and order 5.

Class 6.

Since the primitive groups of class 6 which contain a positive substitution on 6 letters have been determined,* when the groups of class 6 are referred to in what follows we shall have in mind only those which include no positive substitution on 6 letters.

The group G contains by hypothesis a substitution A of the form

$$a_1a_2\,.\,b_1b_2\,.\,c_1c_2.$$

There is a self-conjugate, and hence transitive, subgroup H generated by all the substitutions similar to A. The degree of H is the same as that of G and hence H must displace more than 6 letters. Then some substitution of the series of conjugates $A,\ldots.$ must displace one or more letters new to A. Again one of these generators must connect transitively the letters of A and new letters. If A' is such a substitution, it can not displace more than three letters new to A. Hence G always contains a diedral rotation group of degree 7, 8, or 9. We shall consider these three cases in succession and first merely determine from each of them certain transitive subgroups of H.

The diedral rotation group of degree 7 is itself a primitive group of order 14.

In the second case the group $\{A, A'\}$ is of degree 8 and the substitution AA' is of degree 8 also. This product may be of order 4 or 2.

Let the product AA' be of order 4. The group $J=\{A, A'\}$ is an intransitive octic group. By Theorem I we can find a substitution A'' connecting the two sets of intransitivity of J and introducing at most two new letters. Now G

* MANNING: *loc. cit.*

can not contain two substitutions of class 6 on exactly the same letters, for the product of two such would be a positive substitution of degree 6. Then A'' displaces at least one letter new to J.

If A'' displaces just one new letter, *the group* $K = \{A, A', A''\}$ *is a simply transitive primitive group of degree* 9. The positive subgroup is of class 8 and order 36, so that K has an invariant regular non-cyclic subgroup of order 9.

Again, suppose A'' to introduce two new letters; K is a transitive group of degree 10. Clearly K can not be primitive. Nor can K have two systems of 5 letters each. If there are 5 systems of two letters each, K is isomorphic to the symmetric group on 5 letters and to cycles of three letters in it must correspond positive substitutions of degree 6. This is inadmissible.

The product AA' may be of order 2. Then A and A' are commutative so that A' may be written $a_1b_1 \cdot a_2b_2 \cdot \alpha_1\alpha_2$. There will be another substitution A'' connecting the set $a_1, \ldots.$ with another set, and bringing in one or two new letters. Let A'' bring in just one new letter β. Then we can write at once $A'' = a_1c_1 \cdot a_2c_2 \cdot \alpha_1\beta$; for if two substitutions of class 6 have just three letters in common their product must be of order 3. This means that A'' and A have just four letters in common and their product is of order 2. The group $\{A, A', A''\}$ is of order 12 and has two sets of intransitivity. The substitution A''' can be written either $a_1\alpha_1 \ldots.$ or $a_1\alpha_2 \ldots.$ By the principle just employed it is clear that we get either $A''' = a_1\alpha_1 \cdot b_1\alpha_2 \cdot c_1\beta$ or $A''' = a_1\alpha_2 \cdot b_1\alpha_1 \cdot c_2\gamma$. In the first case we obtain an imprimitive group of order 36 which is a direct product of two symmetric groups, and in the second case a primitive group of order 120 isomorphic to the symmetric group on 5 letters.

Going back to the group $\{A, A'\}$ we see that A'' can not have two new letters in different cycles. For we may write $A'' = a_1c_1 \ldots.$, and since the product AA'' can not be of order 3 it is of degree 8 and order 4 or 2, which is impossible.

It remains to consider the diedral group of degree 9 generated by A and a substitution of the form $a_1a_3 \cdot b_1b_3 \cdot c_1c_3$. Here there are three sets of intransitivity of three letters each. By Corollary I to Theorem I there is a substitution A'' in the series $A, \ldots.$ of the form $a_1b_1 \ldots.$ or $a_1b_2 \ldots.$ It is not necessary to consider $a_1b_3 \ldots.$, since it is the transform of $a_1b_2 \ldots.$ by

$$a_2a_3 \cdot b_2b_3 \cdot c_2c_3.$$

But $\{A, A''\}$ can only be one of the fundamental diedral groups already discussed.

Summing up, we have 4 fundamental transitive groups, one of which must be a subgroup of any primitive group of class 6 not containing positive substitutions on 6 letters. We shall now take up these subgroups in detail.

1. *The* G_{14}^7.

Since all the transitive groups of degree 7 and class 6 are included in the metacyclic group, this G_{14}^7 is not included in a larger primitive group of the same degree. We ask if G_{14}^7 can be the maximal subgroup of a $G_{8.7.2}^8$. The negative substitutions of degree 8 in G^8 must be of order 8. Then among the positive substitutions of G^8 are some of order 4. But the positive subgroup of G^8 is of class 7 and has no operators of order 4.

2. *The* G_{72}^9.

The 12 substitutions of class 6 in G_{72}^9 are made up of the 36 possible combinations of 9 letters, two at a time, so that a G of degree 9 including G_{72}^9 can have no other substitution of class 6. Then G_{72}^9, if contained in a larger group G' of degree 9, is invariant. Let G_1' and G_1 be the subgroups leaving one letter fixed of G' and G_{72}^9 respectively. The subgroup G_1' contains G_1 invariantly, and has negative substitutions other than the 4 of degree 6. They can only be cycles on 8 letters. Then G_1' is transitive and of order 16. If the positive subgroup of G_1' is not the quaternion group, G' contains more than 9 substitutions of degree 8 and order 2, and since from 9 letters only 36 transpositions can be formed, two of these substitutions would have a transposition in common, so that their product is a positive substitution of degree less than 8, an impossibility. Now the quaternion group is completely determined by a subgroup of order 4, so that G_{144}', if it exists, is unique. It does exist and is a well-known group. It may be generated by $12.34.68$, $13.56.78$, $15.47.89$, and 1537.2846. We now show that G_{72}^9 can not be the subgroup leaving one letter fixed in more than one G_{720}^{10}. In G_{720}^{10} there will be a substitution $B = (1\alpha)(..)(..)$ similar to A. Unless B is commutative with the 4 substitutions $24.58.67$, $25.37.69$, $28.39.45$, and $36.48.79$, the commutator of B and one of them will be of degree 7 or less, and positive. Then $B = 1\alpha.25.48$ or $1\alpha.39.67$; but $24.58.67 \times 28.39.45 \times 1\alpha.39.67 = 1\alpha.39.67$, so that G_{720}^{10} is unique. The holomorph of the modular group of degree 10 and order 360 is of order 1440 and is triply transitive on 10 letters. The subgroup leaving three letters fixed is of order 2 and degree 6. The quotient group with respect to the modular

group is the axial group. Corresponding to the three subgroups of order 2 in the quotient group are three subgroups of order 720, one positive of class 8, one positive and negative of class 8, and one positive and negative of class 6. This last group contains G^9_{72}. Since G^9_{72} completely determines G^{10}_{720},* G^{10}_{1440} is completely determined by G^9_{144}. Since neither the modular nor the Mathieu group of degree p^n+1 is maximal in a group of degree p^n+2, this chain of primitive groups ends here.

3. *The Imprimitive* H^9_{36}.

This group is a direct product of two non-Abelian groups of order 6. Its systems of imprimitivity of three letters each can be chosen in only two ways. Any larger group of degree 9 containing H^9_{36} must transform it into itself, since another substitution of class 6 which does not lead back to a previously considered case can not be found. Let G be this larger group and G_1 its subgroup leaving one letter fixed. In G_1 there must be a cycle on 8 letters, which will permute the two substitutions of class 6. Then the square of this cycle transforms a substitution of class 6 into itself. This is an absurdity. Since the systems of imprimitivity of H can not be chosen in 4 ways,† there only remains the possibility that H is the maximal subgroup in a doubly transitive group G on 10 letters of order 360. But this is impossible, for the positive subgroup of order 180 would have 36 subgroups of order 5, and 45 subgroups of order 2, making at once 189 operators.

4. *The* G^{10}_{120}.

If this primitive G_{120} is contained in a larger group G' of the same degree it is invariant in it, since another substitution of class 6 on these letters can not exist. The subgroup G'_1, leaving one letter of G' fixed, transforms G_1, the maximal subgroup of G^{10}_{120}, into itself, and is intransitive in consequence. Since the constituent of degree 3 is symmetric, an operator of G'_1 not in G_1 would lead to another substitution of class 6, which is impossible. Another substitution B, similar to A, introducing one new letter δ, can not be determined subject to the conditions under which $A''' = a_1\alpha_2 \,.\, b_1\alpha_1 \,.\, c_2\gamma$ was determined. Hence we can not pass to a group of degree 11 from G^{10}_{120}.

There are in all 14 primitive groups of class 6, of which 8 contain a substitution of order 3 on 6 letters, and 6 do not. The latter are the G^7_{14}, G^9_{72}, G^9_{144}, G^{10}_{720}, G^{10}_{1440}, and G^{10}_{120}.

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* DE SEGUIER: *Comptes Rendus*, Vol. CXXXVII (1903), p. 37.

† *Transactions of the American Mathematical Society*, Vol. VII (1906), pp. 499-508.

Class 8.

Every group of class 8 contains a substitution $A = a_1a_2 \,.\, b_1b_2 \,.\, c_1c_2 \,.\, d_1d_2$. Any primitive group of class 8 contains a transitive subgroup generated by substitutions similar to A. We shall first determine these subgroups and then take up each one of them in detail, seeking to find in what primitive groups, if any, they may be contained.

If A_1 is a substitution similar to A introducing no new letter, the product AA_1 is of order 4 or 2.

Let $(AA_1)^4 = 1$; then $\{A, A_1\}$ *is a regular octic group.*

Let $(AA_1)^2 = 1$, so that $\{A, A_1\}$ is an axial group. There may or may not be a substitution A_2 similar to A on these same letters. Suppose that A_2 exists and that it is commutative with both A and A_1. Then $\{A, A_1, A_2\}$ *is the regular Abelian group of order* 8, *all of whose substitutions are of order* 2. If A_2 is not commutative with both A and A_1, it must with one of them generate the regular octic group, given above.

We now assume that besides the three substitutions

$$a_1a_2 \,.\, b_1b_2 \,.\, c_1c_2 \,.\, d_1d_2\,, \quad a_1b_1 \,.\, a_2b_2 \,.\, c_1d_1 \,.\, c_2d_2\,, \quad a_1b_2 \,.\, a_2b_1 \,.\, c_1d_2 \,.\, c_2d_1\,,$$

of the axial group, the series of similar substitutions $A, \ldots.$ includes none displacing these 8 letters only. By the Corollary I to Theorem I, the series $A, \ldots.$ contains a substitution A_2 connecting transitively the two sets of letters in $\{A, A_1\}$ and introducing at most three new letters, with not more than one new letter in any cycle, unless there occurs in $A, \ldots.$ a substitution B of the form $a_1\alpha_1 \,.\, a_2\alpha_2 \,.\, b_1\alpha_3 \,.\, b_2\alpha_4$. This exception need not here be considered, since the group $\{B, (BA)^2, (BA_1)^2\}$ is the regular group all of whose operators are of order 2. If A_2 introduces one new letter, $\{A, A_1, A_2\}$ is a transitive group of degree 9, which contains invariantly a regular group of order 9. This is impossible, since the group of isomorphisms of neither of the two groups of order 9 contains an axial subgroup of class 8. Let A_2 introduce two new letters. Then $J = \{A, A_1, A_2\}$ is transitive. It can contain no substitutions of order 3, and hence is of order 40 or 80. Then it has an invariant subgroup of order 5 and degree 10. But the largest group on 10 letters which contains no cycles of 5 letters and which transforms this subgroup into itself is of order 40 and has negative substitutions. If A_2 introduces three new letters, one of the three products $A_2A, \ldots.$ contains a cycle of 4 letters and is of degree 11, an impossibility in a group of class 8.

In what follows it is assumed that the set $A, \ldots.$ contains no two substitutions on exactly the same letters.

For convenience of reference we shall now state some principles of which continual use will be made.

No substitution similar to A can have just one letter in common with A; nor may it have just two letters in common with A unless it is commutative with A. Suppose that B, similar to A, has just three letters in common with A.* The common letters must belong to different cycles in both A and B in order that $(AB)^2$ may not be of class less than 8. But when this condition is satisfied, B is of the form $\alpha_1\alpha_2 . \alpha_3 a_1 . \alpha_4 b_1 . \alpha_5 c_1$ and $(AB)^3$ is of class 4.

If A and B have just 4 letters in common, they generate a positive group, and the product AB can only be of order 4, 3, or 2. The positive intransitive diedral groups in which AB is of degree 12 and orders 10, 8, and 6 are not generated by two substitutions of degree 8. We reject $\{A, B\}$, where $(AB)^4 = 1$, when it is a (4, 1) isomorphism between the intransitive $DG_8^{8,4}$,† itself a simple isomorphism, and a G_2^4, because a $DG_4^{8,8}$ is included. With the product AB still of order 4 and degree 12, $\{A, B\}$ may be a (2, 1) isomorphism between the intransitive $DG_8^{8,6}$ and the intransitive $DG_4^{4,2}$. We may write B, for example, as $a_1b_1 . c_1\alpha_1 . c_2\alpha_2 . \beta_1\beta_2$, and hence $(AB)^2 = a_1b_1 . a_2b_2 . c_1c_2 . \alpha_1\alpha_2$. If AB is of order 3, $\{A, B\}$ is a simple isomorphism between four groups $DG_6^{3,2}$. If B is commutative with A, two cases are to be distinguished as AB is of degree 8 or 12.

If A and B have 5 letters in common, AB is of order 6 and degree 11, or of order 3 and degree 9. When AB is of degree 11, the subgroup $\{A, BAB\}$ is a diedral group of degree 9, class 8, and order 6.

Should the two similar substitutions A and B have just 6 letters in common, the product AB may be of order 5, 4, or 2. The product can not be of order 8, for then $\{A, B\}$ would include the regular octic.

Among the conjugates of A there must exist at least one substitution A_1 which connects old and new letters. The degree of $\{A, A_1\}$ is equal to or less than 12.

Let A_1 introduce just one new letter. If AA_1 is of order 9, $\{A, A_1\}$ *is a transitive diedral rotation group of order* 18.

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* BOCHERT: *Mathematische Annalen*, Vol. XL (1892), p. 176.

† By this notation is indicated a diedral rotation group of degree 8, class 4, and order 8.

The only remaining possibility is that AA_1 is of order 3, in which case $\{A, A_1\}$ is a simple isomorphism between the regular and non-regular representations of the non-Abelian group of order 6. By Theorem I the set $A, \ldots.$ contains a substitution A_2 which connects these two transitive constituents.

Suppose first that A_2 displaces no letter not in $\{A, A_1\}$. Then in the transitive group $\{A, A_1, A_2\}$ any two substitutions of degree 8 and order 2 have just 7 letters in common and their product is of order 3. *Hence* $\{A, A_1, A_2\}$ *is of order* 18 *and contains an invariant non-cyclic group of order* 9.

Let A_2 introduce one new letter. Then $\{A, A_1, A_2\}$ is a primitive group of degree 10. The subgroup leaving one letter fixed must coincide with $\{A, A_1\}$, since it can not contain a subgroup of order 9 without including one of the two transitive groups of order 18 just determined. Hence $G = \{A, A_1, A_2\}$ is of order 60; whence it follows that G contains 6 conjugate subgroups of order 5, by means of which it may be represented as a doubly transitive group of degree 6 and class 4. *This is the icosaedral group which has one and only one primitive representation on* 10 *letters.* We note that the icosaedral group can not be written as an imprimitive group on 10 letters.*

Let $G = \{A, A_1, A_2\}$ be a transitive group of degree 11, with A_2 bringing in two new letters. As before, the order of G is not divisible by 9, so that it is at most $11 \,.\, 10 \,.\, 6 = 660$. That the order is exactly 660, is evident if we note that there must be 12 conjugate subgroups of order 11, each transformed into itself by 55 substitutions. This group, then, if it exists, is simply isomorphic to the doubly transitive group of class 10 on 12 letters, the modular group. Hence the subgroup of order 60 in G is icosaedral, and we are led back to the previous case.

If A_2 introduces 3 new letters, we may assume that it is of the form

$$a'd_3 \,.\, \alpha_1 - \,.\, \alpha_2 - \,.\, \alpha_3 -,$$

where d_3 is the new letter introduced by A_1, and a' is some one of the larger set of intransitivity of $\{A, A_1\}$; α_1, α_2, and α_3 are the 3 new letters. Now A_2 and A have just 4 letters in common and the group $\{A, A_2\}$ is of order 6 with 4 sets of intransitivity. Then A_2 displaces just one letter from each cycle of A. Then A_2 has just 5 letters in common with A_1 or else with A_1AA_1,—with A_1, say. But the product A_1A_2 can not be of order 3, since A_1 and A_2 have no common cycle, nor can it be of order 6 and degree 11, since $\{A_1, A_2\}$ can not have a transitive constituent of degree 2.

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*BURNSIDE: "Theory of Groups" (1897), p. 179.

We now return to A_1 and assume that it displaces two letters not in A. It has been remarked that the product AA_1 can be of order 5, 4, or 2. In the last case only (A and A_1 are commutative) do the two new letters α_1 and α_2 form a cycle of A_1. The group $\{A, A_1\}$* is

$$1, \qquad A_1 = a_1b_1 \,.\, a_2b_2 \,.\, c_1c_2 \,.\, \alpha_1\alpha_2,$$
$$A = a_1a_2 \,.\, b_1b_2 \,.\, c_1c_2 \,.\, d_1d_2, \qquad A_1' = a_1b_2 \,.\, a_2b_1 \,.\, d_1d_2 \,.\, \alpha_1\alpha_2.$$

Let A_2 be a substitution similar to A connecting the set of intransitivity of 4 letters in $\{A, A_1\}$ with one of the other sets. It may be written $a_1c_1 \ldots.$ without loss of generality.

Suppose first that A_2 displaces no letter new to $\{A, A_1\}$. In order that A_2 may not with A, A_1, or A_1' generate one of the diedral subgroups already considered, it must displace α_1, α_2, d_1, d_2, c_1, and c_2. Then A_2 leaves fixed either both b_1, b_2 or both a_2, b_2. It can not leave a_2 and b_1 fixed, for it is not commutative with A_1', as it would then have to be. Then put $A_2 = a_1c_1 \,.\, a_2c_2 \,.\, d_1d_2 \,.\, \alpha_1\alpha_2$; but $A_1'A_2 = a_1b_2c_1 \,.\, a_2b_1c_2$. If A_2 leaves a_2 and b_2 fixed, $A_2 = a_1c_1 \,.\, b_1c_2 \,.\, \alpha_1\alpha_2 \,.\, d_1d_2$ and $A_1'A_2 = a_1b_2c_1 \,.\, a_2c_2b_1$.

Let A_2 bring in one new letter. It may be assumed that $\{A, A_1, A_2\}$ is not transitive. The letters may fall in sets of intransitivity in the following 6 ways; (6, 3, 2), (6, 5), (7, 2, 2), (7, 4), (8, 3), (9, 2). The third and fourth cases may be immediately excluded since they include operators of order 7. In the first case the transitive constituents of degrees 6 and 3 must be in simple isomorphism: hence the constituent on 6 letters is regular and the entire group can not be of class 8. The second case must be a simple isomorphism between two representations of the icosaedral group or else include it. The class is 6. In the fifth case all the substitutions of class 8 will be in an invariant subgroup. In the last case there is an invariant subgroup in the first constituent corresponding to identity of the second. If this subgroup is of class 8, it includes one of the fundamental subgroups already discussed. If it is of class 9, the entire group is of order 6, which is here impossible.

Let A_2 contain two letters β_1 and β_2, not displaced by $\{A, A_1\}$. We may write $A_2 = a_1\alpha_1 \ldots.$ Suppose first that A_2 leaves α_2 fixed. Then $\{A, A_2\}$ is of degree 11 and order 6, and A_2 has a cycle in common with A. Comparing A_2 with A_1 and A_1', this is seen to be impossible. If A_2 displaces α_2, AA_3 is of order

* This group need not be considered, as we might restrict A_1 to substitutions which connect old and new letters, but it is found convenient to use it.

3 or 4, since $\{A, A_2\}$ is of degree 12. In the first case $\{A_1, A_2\}$ is of degree 11. But this can not be admitted here. In the second case $A_2 = a_1\alpha_1 . a_2\beta_1 . x_1x_2 . \alpha_2\beta_2$, where x_1 and x_2 are letters from two cycles of A. Since $\{A_1, A_2\}$ can not be of degree 9, 11, or 12, it is of degree 10, and $x_1x_2 = b'c'$, so that $\{A_1', A_2\}$ is of degree 11, which is clearly inadmissible.

Let A_2 introduce three new letters. We again write it $a_1\alpha_1 \ldots$. Since A and A_2 can not have just 3 letters in common, A_2 can not displace α_2. It follows that AA_2 is of degree 12 and order 3, so that A_2 displaces one letter from each cycle of A. Then $\{A_1, A_2\}$ can be neither of degree 11 nor 12.

If the product AA_1 is of order 4, the preceding axial group is included in the group which A and A_1 generate.

The only remaining case when A_1 brings in two new letters is that AA_1 is of order 5 and degree 10. We write down the 5 substitutions of order 2 in $\{A, A_1\}$:

$$\begin{aligned} A &= a_1a_2 . b_1b_2 . c_1c_2 . d_1d_2, & A_1 &= a_1b_1 . b_2\alpha_1 . c_1d_1 . d_2\alpha_2, \\ A_1' &= a_1b_2 . a_2\alpha_1 . c_1d_2 . c_2\alpha_2, & A_1'' &= a_1\alpha_1 . a_2b_1 . c_1\alpha_2 . c_2d_1, \\ A_1''' &= a_2b_2 . b_1\alpha_1 . c_2d_2 . d_1\alpha_2. \end{aligned}$$

As before, it may be assumed that there is a substitution similar to A connecting the two transitive constituents of $\{A, A_1\}$ which is of the form $A_2 = a_1c' \ldots$, where c' is one of the 5 letters of the second set of intransitivity.

A transitive group of degree 10 can not now include a subgroup of degree 9. But it must have 6 conjugate subgroups of order 5. Hence, if there is a group in this case, it is icosaedral, and has been already mentioned. If A_2 brings in one new letter, $\{A, A_1, A_2\}$ is of degree 11, and as before can only exist if it includes the (primitive) icosaedral group of degree 10.

Let A_2 introduce two new letters. We may write $A_2 = a'c' . \alpha_1'\beta_1 . \beta_2 \ldots$, where a' and c' are from different systems of $\{A, A_1\}$, and α' is one of the two letters α_1, α_2. Clearly $\{A, A_2\}$ can not be of degree 11, nor can it be of degree 12 and order 4, for then it would include the axial subgroup of degree 10, which has been considered. The only possibility is that AA_2 is of order 2 and degree 12. Then $A_2 = a'c' . a''c'' . \alpha_1\beta_1 . \alpha_2\beta_2$. The group $\{A, A_1, A_2\}$ is imprimitive and its 6 systems are permuted according to a primitive group, positive and of class 4, that is, according to the icosaedral group of degree 6. A substitution of $\{A, A_1, A_2\}$ which does not permute the systems of imprimitivity can not be of degree 8, for then it would be on the same 8 letters as some similar substitution

of the group. Then $\{A, A_1, A_2\}$ can not be of order 240, and since it can not be of order 60, its order is certainly 120. Now A_2 is either $a_1c_2 \,.\, a_2c_1 \,.\, \alpha_1\beta_1 \,.\, \alpha_2\beta_2$ or $b_1d_2 \,.\, b_2d_1 \,.\, \alpha_1\beta_1 \,.\, \alpha_2\beta_2$. But the two groups generated by $\{A, A_1, A_2\}$ and one of the above substitutions are conjugate under $a_1b_1 \,.\, a_2b_2 \,.\, c_1d_1 \,.\, c_2d_2 \,.\, \alpha_1\alpha_2 \,.\, \beta_1\beta_2$. Hence this case leads to but *the one well-known imprimitive* $G_{120}^{12,\,8}$.

If three new letters are brought in by A_2, we may write it

$$a'c' \,.\, \alpha'\beta_1 \,.\, \beta_2 — \,.\, \beta_3 —.$$

Since $\{A, A_2\}$ can be neither of degree 12 nor 13, this case goes out at once.

When A_1 generates with A a group of degree 11, there is only one case to consider: $\{A, A_1\}$ is of order 6 and the 3 substitutions of class 8 are

$$A,\ A_1 = a_1\alpha_1 \,.\, b_1\alpha_2 \,.\, c_1\alpha_3 \,.\, d_1d_2, \qquad A_1' = a_2\alpha_1 \,.\, b_2\alpha_2 \,.\, c_2\alpha_3 \,.\, d_1d_2.$$

It should now be noted that the product of any two non-commutative substitutions similar to A is of order 3. The product, if the substitutions are commutative, is of degree 12 or 16. The transitive group to which this subgroup leads can not contain a substitution B, similar to A, of the form $a_1d_1 \ldots$, for $B = a_1d_1 \,.\, a_2d_2 \ldots.$ and also $B = a_1d_1 \,.\, \alpha_1d_2 \ldots$, a contradiction. This is true if any other letter of $\{A, A_1\}$ except d_2 be taken for a_1. Again B can not have the cycle $(d_1\delta)$ for then the three products AB, A_1B, and $A_1'B$ would be of order 3, an impossibility. But in order to obtain a transitive group generated by substitutions of the series $A, \ldots$, we must finally get a substitution $a'd \ldots.$ similar to A, where a' is a letter of A.

Let A_1 displace 4 letters new to A, and let the group

$$\{A,\ A_1 = a_1b_1 \,.\, a_2b_2 \,.\, \alpha_1\alpha_2 \,.\, \beta_1\beta_2\},$$

which we shall call J, be taken up first. By Theorem I there is, in the series $A, \ldots$, a substitution connecting a_1 with some letter of J not in the first set. We may assume this substitution to be $A_2 = a_1c_1 \,.\, a_2c_2 \,.\, \alpha_1\gamma_1 \,.\, \alpha_2\gamma_2$, where γ_1 and γ_2 are new to J. The group $K = \{J, A_2\}$ is of order 12 and contains one other substitution of class 8: $A_2' = b_1c_1 \,.\, b_2c_2 \,.\, \alpha_2\gamma_1 \,.\, \beta_2\gamma_2$.

We have to consider only 4 substitutions which connect the set $a_1 \ldots.$ of K with other sets:

$$\begin{aligned}
&(1) \quad B_1 = a_1\alpha_2 \,.\, b_1\alpha_1 \,.\, c_2\delta \,.\, \gamma_2d_1,\\
&(2) \quad B_2 = a_1\alpha_1 \,.\, c_1\gamma_1 \,.\, b_1\alpha_2 \,.\, d_1\delta,\\
&(3) \quad B_3 = a_1d_1 \,.\, a_2d_2 \,.\, \alpha_2\gamma_2 \,.\, \gamma_1\beta_2,\\
&(4) \quad B_4 = a_1d_1 \,.\, a_2d_2 \,.\, \alpha_1\delta_1 \,.\, \beta_1\delta_2.
\end{aligned}$$

The substitutions B_1 and B_2 connect the sets to which a_1 and α_1 belong.

The group $L = \{K, B_1\}$ is of order 120, and is a simple isomorphism between a $G_{120}^{5,2}$ and a $G_{120}^{10,6}$. The remaining 6 substitutions of class 8 are

$$a_1\gamma_1 \,.\, b_2\delta \,.\, c_1\alpha_1 \,.\, d_1\beta_2, \quad a_1\delta \,.\, b_2\gamma_1 \,.\, c_2\alpha_2 \,.\, d_2\beta_1,$$
$$a_2\gamma_1 \,.\, b_1\delta \,.\, c_2\alpha_1 \,.\, d_2\beta_2, \quad a_2\alpha_2 \,.\, c_1\delta \,.\, b_2\alpha_1 \,.\, d_2\gamma_2,$$
$$a_2\delta \,.\, b_1\gamma_1 \,.\, c_1\alpha_2 \,.\, d_1\beta_1, \quad \gamma_1\alpha_2 \,.\, b_1c_1 \,.\, b_2c_2 \,.\, \beta_2\gamma_2.$$

The subgroup of L leaving the letter a_1 fixed is intransitive in sets $d_1\beta_1$, $d_2\beta_2\gamma_2$, etc. Then it may be assumed that the substitution which connects the two sets of L is either

$$C_1 = a_1d_1 \,.\, a_2d_2 \,.\, \alpha_2\gamma_2 \,.\, \gamma_1\beta_2,$$

or

$$C_2 = a_1d_2 \,.\, a_2d_1 \,.\, \beta_1\delta \,.\, \alpha_1\varepsilon.$$

Now $\{L, C_1\}$ *is a primitive group of order* 720 *isomorphic to* (abcdef) all. In the group $\{L, C_2\}$ the conjugates of the product

$$\gamma_1\alpha_2 \,.\, b_1c_1 \,.\, b_2c_2 \,.\, \beta_2\gamma_2 \,.\, a_1d_2 \,.\, a_2d_1 \,.\, \beta_1\delta \,.\, \alpha_1\varepsilon$$

generate a self-conjugate regular Abelian subgroup of order 16, of type (1, 1, 1, 1). *This fundamental group can readily be identified with the primitive $G_{1920}^{16,8}$ given by* MILLER.*

We return to the group K. The group $\{K, B_2\} = L$ is of order 36 and has one set of intransitivity of degree 9, and two of degree 3. There is one other substitution of class 8: $a_2\alpha_1 \,.\, c_2\gamma_1 \,.\, b_2\alpha_2 \,.\, d_2\delta_1$. Consider now the two possible substitutions

$$C_1 = a_1\beta_1 \,.\, b_1\beta_2 \,.\, c_1\gamma_2 \,.\, d_1\varepsilon,$$

and

$$C_2 = a_1d_1 \,.\, a_2d_2 \,.\, \alpha_1\delta \,.\, \beta_1\varepsilon.$$

Since $s = \beta_1d_1 \,.\, \beta_2d_2 \,.\, a_2b_1 \,.\, c_1\alpha_1 \,.\, c_2\alpha_2 \,.\, \gamma_2\delta$ gives $sC_1s = C_2$ and $sLs = L$, only C_1 need be used. One constituent of $M = \{L, C_1\}$ is the symmetric group of degree 4 and the other is an imprimitive group on 12 letters, whose 4 systems are permuted according to the symmetric group. The order of M is 144. The new substitutions of class 8 are: $a_2\beta_1 \,.\, b_2\beta_2 \,.\, c_2\gamma_2 \,.\, d_2\varepsilon$ and $\alpha_1\beta_1 \,.\, \gamma_1\gamma_2 \,.\, \alpha_2\beta_2 \,.\, \delta_1\varepsilon$. There is only one form in which D_1 may be assumed:

$$D_1 = a_1d_1 \,.\, a_2d_2 \,.\, \alpha_1\delta \,.\, \beta_1\varepsilon.$$

The group $H = \{M, D_1\}$ is imprimitive of degree 16 *and order* 576. It is the direct product of two symmetric groups of order $\underline{|4}$.

*MILLER: AMERICAN JOURNAL OF MATHEMATICS, Vol. XX (1898) p. 229.

It may now be assumed that the series $A, \ldots$ contains no substitution which connects the set $a_1 \ldots$ of K with $\alpha_1 \ldots$ or $\beta_1 \ldots$. Let us consider $\{K, B_3\}$. This group has two sets of intransitivity, so that a substitution of the form $a_1 x \ldots$ exists, where x is a letter of the set $\alpha_1 \ldots, \beta_1 \ldots,$—contrary to hypothesis. Finally $\{K, B_4\}$ leads uniquely to the substitution $a_1\delta_1 . d_1\alpha_1 . b_2\gamma_2 . c_2\beta_2$, which connects the sets $a_1 \ldots$ and $\beta_1 \ldots$.

In conclusion we have to consider $A_1 = a_1\alpha_1 . b_1\alpha_2 . c_1\alpha_3 . d_1\alpha_4$, and the group $\{A, A_1\}$. But by Corollary II to Theorem I, we know that there is in the series $A, \ldots$ a substitution $A_2 = a_1 b' \ldots$, where b' is one of the 3 letters b_1, b_2, α_2. Hence this case is included in one of the preceding.

Nine distinct transitive groups have been determined, one of which is included in any primitive group of class 8. We shall take up these subgroups one by one and find what primitive groups are determined by them. In discussing a particular subgroup use may again be made of any special theorems or processes employed in obtaining it originally.

1. *The Regular Octic.*

Our primitive group G contains the regular octic (I). Since $A, \ldots$ generate a transitive group of degree greater than 8, there is a substitution B similar to A which connects some new letters transitively with the letters of this octic group. Now $\{I, B\}$ can not be a transitive group of degree 9, for I is not a subgroup of the group of isomorphisms of a group of order 9. If I is contained in a primitive group of degree greater than 9, it is contained in an imprimitive group of degree 10 and order 80. That it can not be contained in an imprimitive group of degree 12, maximal in a primitive group, results from the fact that of the three subgroups of order 4 in I one is characteristic, so that there are but two interchangeable systems of 4 letters each with one letter in common. Now the group of order 80 can not have an invariant subgroup of order 5 since the largest group of degree 10 and class 8 in which a subgroup of order 5 is invariant is of order 40. Nor, with 35 substitutions displacing 8 letters, can there be 16 subgroups of order 5 and degree 10.

2. *The Regular Abelian G_8^8 of Type* (1, 1, 1.)

This case is based upon the regular group (I) all of whose substitutions are of order 2. It may be written:

$$\begin{array}{ll} 1, & A_1 = 12 \,.\, 34 \,.\, 56 \,.\, 78, \\ A_2 = 13 \,.\, 24 \,.\, 57 \,.\, 68, & A_3 = 14 \,.\, 23 \,.\, 58 \,.\, 67, \\ A_4 = 15 \,.\, 26 \,.\, 37 \,.\, 48, & A_5 = 16 \,.\, 25 \,.\, 38 \,.\, 47, \\ A_6 = 17 \,.\, 28 \,.\, 35 \,.\, 46, & A_7 = 18 \,.\, 27 \,.\, 36 \,.\, 45. \end{array}$$

Since the group of isomorphisms of the non-cyclic group of order 9 does not include I, the new substitution B, connecting new letters transitively with the letters of I, can not introduce just one new letter.

If B introduces just two new letters, $\{I, B\}$ is imprimitive of order 80. There are 35 substitutions of class 8 and 44 of degree 10, the latter in 11 subgroups of order 5, since $\{I, B\}$ is positive. But 11 is not a divisor of 80.

If B brings in three new letters, and has not already two new letters in a cycle, i can be chosen so that BA_iB has one cycle of new letters. Then $\{I, B\}$ includes a subgroup of degree 9 and class 8, an impossibility.

Let B ntroduce 4 new letters. The new letters must be in different cycles of B. The 4 old letters of B must unite by twos in three of the substitutions of I and must be in different cycles in the remaining 4. Since the holomorph of I is triply transitive, we can write $B = 1\alpha \,.\, 2\beta \,.\, 3\gamma \,.\, 4\delta$, the element 4 being fixed by the remark just made. The group $J = \{I, B\}$ if of order 96, is 3-fold imprimitive on 6 systems. It can not be contained in a primitive group G of degree 12. Nor can a substitution C similar to A be found introducing one new letter. We would have $C = 1x \,.\, —$ containing three of the letters $\alpha, \beta, \gamma, \delta$, and three of the letters 5, 6, 7, 8, as is seen by comparing with $B'IB'$, where $B' = BA_4B$. Then $CA_1C = x2 \,.\, 34 \,.\, 5'\alpha' \,.\, \beta'\gamma'$ and $(A_1C)^2$ is of degree 8 or 10 with the cycle $(1x2)$, which is impossible. It is evident that G can not contain a substitution or subgroup on 13 letters, so that if C introduces two new letters $\{I, C\}$ is imprimitive with 7 systems. Since the group according to which these systems are permuted can not be of class less than 4 and certainly is of class 4, the systems go among themselves according to the simple $G^{7,4}_{168}$. In the three ways of dividing the letters of J^{12}_{96} into systems α is associated with β, γ, δ respectively. The systems of $K = \{J, C\}$ must conform to one of these three different groupings. Let us suppose for the moment that α, β form a system of imprimitivity in K. Then the substitution C can be written $\alpha x \,.\, \beta y \ldots$, since K is doubly transitive in the systems. Further, since I is transitive, $C = \alpha x \,.\, \beta y \,.\, 1 — \,.\, 2 —$, if we choose. Comparing C with $1\alpha \,.\, 2\beta \,.\, 3\gamma \,.\, 4\delta$ it is seen that the two remaining elements belong to 5, 6, 7, 8. Then

$$C = \alpha x \cdot \beta y \cdot \left\{ \begin{array}{c} 15 \cdot 26 \\ 16 \cdot 25 \\ 17 \cdot 28 \\ 18 \cdot 27 \end{array} \right\}.$$

The last three forms of C are the transforms of the first by

$$\alpha\beta \cdot xy \cdot \gamma\delta \cdot \left\{ \begin{array}{c} 56 \cdot 78 \\ 57 \cdot 68 \\ 58 \cdot 67 \end{array} \right\}$$

respectively, and these substitutions transform J into itself. Similarly the groups generated by J and

$$\alpha x \cdot 15 \cdot \left\{ \begin{array}{c} \beta y \cdot 26 \\ \gamma y \cdot 37 \\ \delta y \cdot 48 \end{array} \right\}$$

are conjugate under $\beta\gamma\delta \cdot 678 \cdot 234$. Then $K = \{J, \alpha x \cdot \beta y \cdot 15 \cdot 26\}$. Here the systems of imprimitivity can be chosen in only one way, so that K is contained in a doubly transitive group of degree 15. Then we can find a substitution D similar to A introducing one letter θ new to K. Since K, J, I are all transitive, we can write $D = \theta x \cdot \alpha\omega_1 \cdot 1\omega_2 \cdot \omega_3\omega_4$. Comparing D with I we see that ω_1 can not be one of the letter of I, and that ω_2 must be displaced by I. One of the substitutions of I contains the cycle $1\omega_2$ and must be commutative with D. If ω_3 and ω_4 are not also elements of I, some commutator $(A_iD)^2$ is of class equal to or less than 6. Then $\omega_2\omega_4$ is also a cycle of $1\omega_2 \ldots$ of I. Now comparing D with the substitutions of I', the transform of I by $5\alpha \cdot 6\beta \cdot 7\gamma \cdot 8\delta$, it becomes evident that $\omega_1 = \beta$, and $\omega_2 = 2$, or $\omega_1 = \gamma$ and $\omega_2 = 3$, or $\omega_1 = \delta$ and $\omega_2 = 4$. Comparing D with I'', the transform of I by $1\alpha \cdot 2\beta \cdot 3\gamma \cdot 4\delta$, we see that ω_3 and ω_4 are chosen from the elements 5, 6, 7, 8. If $\omega_1 = \beta$, the commutator of $D = \theta x \cdot 12 \cdot \alpha\beta \cdot \omega_3\omega_4$ and $CBA_1BC = 12 \cdot 78 \cdot \gamma\delta \cdot xy$ is of class less than 8. If $D = \theta x \cdot \alpha\gamma \cdot 13 \cdot 68$ or $= \theta x \cdot \alpha\delta \cdot 14 \cdot 67$, $(DCBA_2BC)^6$ is of degree 4. There remain only two possible forms for D:

$$\theta x \cdot a\gamma \cdot 13 \cdot 57 \quad \text{and} \quad \theta x \cdot \alpha\delta \cdot 14 \cdot 58.$$

These two substitutions are conjugate under $\gamma\delta \cdot 34 \cdot 78$, a substitution which transforms the group K into itself. Hence we can assume $D = \theta x \cdot \alpha\gamma \cdot 13 \cdot 57$ and the group $G = \{K, D\}$ is unique. The order of G is at least $8\,!/2$. The Abelian group of order 16 and type (1, 1, 1, 1) generated by

$$12 \,.\, 34 \,.\, 56 \,.\, 78 \,.\, \alpha\beta \,.\, \gamma\delta \,.\, xy \,.\, \theta\eta, \quad 13 \,.\, 24 \,.\, 57 \,.\, 68 \,.\, \alpha\gamma \,.\, \beta\delta \,.\, x\eta \,.\, y\theta,$$
$$15 \,.\, 26 \,.\, 37 \,.\, 48 \,.\, \alpha\eta \,.\, \beta\theta \,.\, \gamma x \,.\, \delta y, \quad 1\alpha \,.\, 2\beta \,.\, 3\gamma \,.\, 4\delta \,.\, 5\eta \,.\, 6\alpha \,.\, 7x \,.\, 8y$$

is transformed into itself by G, and hence G is the group of isomorphisms of this Abelian group, whose order is $8!/2$. This group is known to be isomorphic to the alternating group on 8 letters.

If G were contained in a larger primitive group G' of degree 15, G'_1, the subgroup of G' leaving θ fixed would contain G_1 self-conjugately. The subgroup G'_1 must be imprimitive on 7 systems which it permutes according to a group of class 4, that is, the $G_{168}^{7,4}$. Then G'_1 contains a new substitution S of order 2 transforming each system into itself. The intransitive head F of K contains two substitutions on 4 systems having in common any two arbitrary systems. If S is of degree 8 or 10 we can find two substitutions in F having two cycles in common with it, and hence one of them has three in common with S, so that when multiplied into S it gives a product of class 4 or 6. If S has 6 or 7 cycles, F contains a substitution having 4 cycles in common with it. Then G'_1 coincides with G_1 and G' with G.

We now pass to a G of degree 16. Recalling the determination of D, it is clear that E can be put uniquely in the form $\eta x \,.\, \alpha\delta \,.\, 14 \,.\, 58$. The conjugates of $(34)(78)(\alpha\beta)(xy) \times E$ generate the regular group of order 16 all of whose operators are of order 2, and hence the new group is the holomorph of this regular Abelian group.

3. *The* $D_{18}^{9,8}$.

In this and all the following cases no two substitutions of order 2 can displace exactly the same 8 letters.

Since the group of isomorphisms of the cyclic group of order 9 is the cyclic group of order 6 and class 6, this DG_{18}^{9} can not be contained in a larger group of the same degree and class 8.

Since the DG_{18}^{9} is only one-fold imprimitive, it is contained in a doubly transitive group of degree 10 and order 180, if contained in a primitive group at all. There are at least 9 conjugate subgroups of order 5 in G_{180}^{10}, and hence by Sylow's theorem as many as 36. This is impossible. This case gives no group.

4. *The Imprimitive $G_{18}^{9, 8}$ in Which the Characteristic Subgroup is Non-Cyclic.*

We write out this group I:

1,	123 . 455 . 789,	132 . 465 . 798,
147 . 369 . 258,	174 . 396 . 285,	159 . 267 . 348,
195 . 276 . 384,	168 . 249 . 357,	186 . 294 . 375,
23 . 86 . 47 . 59,	31 . 49 . 58 . 67,	12 . 57 . 69 . 48,
56 . 92 . 71 . 83,	64 . 73 . 82 . 91,	45 . 18 . 93 . 72,
89 . 35 . 14 . 26,	97 . 16 . 25 . 34,	78 . 24 . 36 . 15.

In this group the systems of imprimitivity of three letters each can be chosen in 4 different ways.

We first seek the primitive groups of degree 9 which contain this transitive subgroup I invariantly. These groups are in the holomorph of the non-cyclic groups of order 9. There are three groups of class 8 here, the $+ G_{36}^{9}$, the $+ G_{72}^{9}$, with a quaternion subgroup, and the $\pm G_{72}^{9}$, with a cyclic subgroup of order 8. The G_{36}^{9} is invariant in the last two, which are both doubly transitive.

Next let G be any primitive group of degree 10 containing I. The group G is at least doubly transitive, and can contain no imprimitive invariant subgroup;* that is, $H = \{A, \ldots\}$ is also at least doubly transitive. The subgroup G_1, which leaves one letter of G fixed, can not be of order 18 for the same reasons as in the preceding section. If G_1 is of order 36, G_2 is cyclic of order 4, and hence G_1 is the primitive group of this order just mentioned. Now G_1 completely determines the modular group of order 360 and degree 10, which in turn is not maximal in a group of degree 11.†

The $\pm G_{72}^{9}$ completely determines the Mathieu group of order 720 on 10 letters. This leads to no group on 11 letters.

If a group of degree 10 and order 720 exists containing the $+ G_{72}^{9, 8}$, it contains the $G_{360}^{10, 8}$ invariantly. For the subgroup leaving two letters fixed is transformed into itself by a substitution similar to A transposing the two letters left fixed by G_{8}^{8}. This substitution will permute at most two of the subgroups of order 4 in the quaternion group and transforms one into itself. Then with G_{36}^{9} it generates the G_{360}^{9}. Now the holomorph of the $G_{360}^{10, 8}$ is well-known.‡ One of

* MANNING: *Bulletin of the American Mathematical Society*, Vol. XIII, Ser. 2 (1906), p. 23.

† DE SEGUIER: *Comptes Rendus*, Vol. 137 (1903), p. 37.

‡ BURNSIDE: "Theory of Groups," (1897), p. 248.

its three subgroups of order 720 includes the $+ G_{72}^{9,8}$. If a transitive group of degree 11 and order 7920 exists, it is the 4-ply transitive group of Mathieu and leads in turn to the 5-ply transitive group of order 12 . 11 . 10 . 8 . 9.*

We now return to the fundamental group of order 18. Let $A_1, \ldots, A_9$ be its substitution of order 2 in order as written. It may be assumed that there is a substitution B similar to A_1, connecting some new elements with the elements of I, and such that $\{B, I\}$ does not include one of the primitive groups we have already.

If B introduces one new letter, $\{I, B_1\}$ is of degree 10 and gives nothing new. The same is true if B brings in just two new letters. If B connects 3 new letters α, β, γ with the old letters of I, we must write $B = 1\alpha \cdot \omega_1\beta \cdot \omega_2\gamma \cdot \omega_3\omega_4$. Since A_1 leaves the element 1 fixed, $BA_1B = \beta\gamma \ldots$, so that $\omega_1\omega_2$ is a cycle of A_1, and ω_3 and ω_4 are in different cycles of A_1. The subgroup of $G_{432}^{9,6}$ leaving the element 1 fixed is the group of isomorphisms of G_9^9 of type (1, 1) and in it A_1 is invariant. Then I is invariant under this G_{48}^8, leaving 1 fixed. The cycles of A_1 are the systems of imprimitivity of G_{48}. Further, the subgroup of G_{48} leaving the system 23 fixed is transitive. Then B can be written $1\alpha \cdot 2\beta \cdot 3\gamma \cdot 4\left\{\begin{smallmatrix}5\\6\end{smallmatrix}\right\}$. Again, the substitution $23 \cdot 56 \cdot 89 \cdot \beta\gamma$ transforms I into itself and finally fixes the substitution $B = 1\alpha \cdot 2\beta \cdot 3\gamma \cdot 45$. The group $J = \{I, B\}$ has 4 systems of imprimitivity which are permuted according to a transitive group of class 2 containing a substitution of order 3; that is, according to the symmetric group on 4 elements. Since $(A_1B)^2 = 23 \cdot 45 \cdot 79 \cdot \beta\gamma$ does not permute the systems, the subgroup which leaves all the systems fixed is of order 18 and J is of order 432. We pass from J to a group of degree 13. Since the systems of J can be chosen in only one way, it is contained in a doubly transitive group on 13 letters. Since in this $G_{5616}^{13,8}$ there are just 13 conjugate subgroups of order 18 similar to I, each subgroup is transformed into itself by a group of order 432, having one transitive constituent of degree 4 and order 24. The other constituent is the largest group on 9 letters which transforms I into itself. Hence $G_{5616}^{13,8}$ is completely determined by its subgroup leaving one letter fixed. This group of order 5616 is the doubly transitive group according to which the 13 subgroups of order 3 in the Abelian G_{27} of type (1, 1, 1) are permuted by its group of isomorphisms. This case is completed.

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* MATHIEU: *Liouville's Journal*, Ser. 2, Vol. VI (1861), p. 274. JORDAN: *Liouville's Journal*, Ser. 2, Vol. XVII (1872), p. 351.

5. *The Icosaedral* $G_{60}^{10,\,8}$.

This fundamental group is primitive. It can not be contained in a larger group of the same degree and class. It leads to a doubly transitive group on 11 letters, simply isomorphic to the modular group $G_{660}^{12,\,10}$. There is but one such group. Since in a $G_{7920}^{12,\,8}$ in which $G_{660}^{10,\,8}$ is maximal $G_{60}^{10,\,8}$ must be transformed into itself by 120 substitutions, the triply transitive group of order 7920 is unique. It is isomorphic to the 4-ply transitive group of degree 11.

6. *The Imprimitive* G_{120}^{12}.

This group is (2, 1)-isomorphic to $(abcde)+$. Suppose G_{120}^{12} to be included in an imprimitive G of degree 12. The presence of substitutions of degree 10 and order 5 requires that there be either 2 or 6 letters in each system of imprimitivity of the larger group. If there are just two systems, G has an intransitive subgroup of half its order. Then the letters of G_{120}^{12} can be arranged in two systems of 6 each, such that an intransitive head of order 60 is included in it. This head is a simple isomorphism between two icosaedral groups. In this head are included the 15 substitutions of class 8 of G_{120}^{12}, which contradicts the fact that the substitutions of class 8 generate the whole group. Now G_{120}^{12} can be included invariantly in a G_{240}^{12} in which the systems are permuted according to a symmetric group of order 120 written on 6 elements. But G_{120}^{12} can not be contained in a primitive group G of degree 12. For G can not be doubly transitive, nor, as we shall show, can it be simply transitive. In the latter case G_1 would be a simple isomorphism between two transitive groups of degrees 5 and 6 respectively. Now the only transitive groups of degree 5 which can be represented transitively on 6 letters include the icosaedral. This isomorphism is of class 6. We remark that the systems of imprimitivity of G_{120}^{12} can be chosen in only one way. Then G_{120} may be contained in a primitive group of degree 13. In this larger group a particular subgroup of order 2 and degree 8 is transformed into itself by 40 or 80 operators. But the largest group on the same letters transforming this substitution into itself is of order $2^4 \,.\, 4!$ and includes no operator of order 5.

7. *The Primitive* G_{720}^{15}.

We now consider the primitive $G_{720}^{15,\,8}$. The subgroup G_1 leaving one letter fixed is of order 48 and is a (2, 1) isomorphism between two transitive constituents of degrees 8 and 6 respectively. If G is invariant in a larger simply transitive

group G' of degree 15, G_1 is invariant in G'_1, and the latter has the same sets of intransitivity as G_1. Now the head of the first (imprimitive) constituent in G_1 is the symmetric group of order 24, a complete group, so that any group in which it is invariant is a direct product. But the tail of this constituent already contains all the substitutions commutative with each operator of the head. Nor can G_{720} be invariant in a doubly transitive group of degree 15, for the only other substitution of class 8 we can determine is $B = 18 . 26 . 35 . ex$, bringing in one new letter x. Now $\{G, B\}$ has a regular invariant subgroup of order 16, and is of order 11,520. Any other substitution of class 8 we may determine leads again to B.

8. *The Primitive* $G_{1920}^{16,8}$.

If another substitution of class 8 and order 2 displacing no new letters occurs in a larger group containing $G_{1920}^{16,8}$, we shall have either $12 \ldots$ or $1a \ldots$. If it is $12 \ldots$, we continue and see that it is $12 . 68 . 79 \ldots$, an impossibility. If it is $1a \ldots$, we get uniquely $1a . 6d . 7e . 5c$. The substitution $3a . 6b . ae . 8c$ transforms $\{G_{1920}^{16}, 1a . 6d . 7e . 5c\}$ into $G_{11520}^{16,8}$.

Since the subgroup G_1 of G_{1920} is a complete group, G can not be invariant in a larger primitive group.

9. *The Imprimitive* $H_{576}^{16,8}$.

The fundamental transitive subgroup in this case is the imprimitive direct product of order $(24)^2$. Suppose this group is invariant in a primitive group G. Now H has a characteristic regular subgroup of order 16 and H_1 has a characteristic subgroup of order 9. Since the non-cyclic group of order 9 admits no isomorphism of period 5 or 7, G contains no operator of order 5 or 7. Then G_1 is of order $36 . 2$ or $36 . 4$ and can not be transitive. The transitive constituent of degree 9 in H_1 must be transformed into itself by G_1. We determine uniquely a substitution $B = a_2 b_1 . c_1 \alpha_1 . c_2 \alpha_2 . d_1 \beta_1 . d_2 \beta_2 . \gamma_2 \delta$, which gives us the primitive $G_{1152}^{16,8}$. The group G_{1152} is not in turn invariant in a larger group. If on the other hand H_{576} is not invariant in a larger group, there must be another substitution of class 8 and order 2. It is uniquely determined as

$$B = a_1 d_2 . a_2 d_1 . \gamma_1 \beta_2 . \gamma_2 \alpha_2 .$$

This group $\{H, B\}$ is a conjugate of G_{11520}^{16}.

We have obtained the following 18 primitive groups of class 8 :

$$G^{9}_{36},\quad G^{9}_{72},\quad \pm G^{9}_{72},\quad G^{10}_{60},\quad G^{10}_{360},\quad \pm G^{10}_{720},$$
$$G^{10}_{720},\quad G^{11}_{660},\quad G^{11}_{7920},\quad G^{12}_{7920},\quad G^{12}_{12\,.\,7920},\quad G^{13}_{5616},$$
$$G^{15}_{720},\quad G^{15}_{8!/2},\quad G^{16}_{1152},\quad G^{16}_{1920},\quad G^{16}_{11520},\quad G^{16}_{16(8!/2)}.$$

In JORDAN'S enumeration of the primitive groups of class less than 14, he mentions the groups of degree $\frac{K(K-1)}{2}$ formed by the displacements to which the symmetric (alternating) groups on K letters $a, b, c \ldots .$ subject the $\frac{K(K-1)}{2}$ binary products $ab, ac, \ldots ..$ He states that these groups belong to the class $2K-2$ (to the class $3K-3$), and are primitive if $K > 4$. These formulae for the class are incorrect. For this representation of the symmetric group the class is $2K-4$, and for the alternating $4K-12$ when $K \overline{\overline{<}} 6$, $3K-6$ when $K \overline{\overline{>}} 6$.

A consequence of these formulae is that a simply transitive primitive group can always be found for which the ratio of the degree to the class is greater than any given number.

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